the capacitance between two conductors, the effect of a third conductor is the same as that of a magnetic wall. However, there might be some additional capacitance between each conductor and the third one.

To demonstrate the use of the equivalent length concept, let us take the case of Cristal's [7] filter which was originally designed for a frequency of 1.5 GHz. Let us consider Cristal's resonators (fingers) 3 and 4 and their center-to-center spacing as pitch p. For the filter

$$\begin{aligned} C_0 &= 4.5\epsilon_0 \\ C_\pi &= 5.372\epsilon_0 \\ C_{\pi/2} &= 4.93\epsilon_0 = 0.43684 \text{ pF/cm} \\ Z_0 &= 76.38 \text{ ohms} \end{aligned}$$

(width of equivalent zero thickness conductor) $w = 0.438$ inch. For C_e,

$$\begin{aligned} C_{f_0}' &= 0.514\epsilon_0 \\ C_e &= 2wC_{f_0}' = 0.1012 \text{ pF}. \end{aligned}$$

For C_{f_2},

$$\begin{aligned} C_{f_e}' &= 0.3516\epsilon_0 \\ C_{f_1} &= 2wC_{f_e}' = 0.06927 \text{ pF}. \end{aligned}$$

For C_{f_2},

$$\begin{aligned} C_{f_e}' &= 0.350\epsilon_0 \\ wC_{f_e}' &= 0.03448 \text{ pF} \\ C_t &= C_e + C_{f_1} + C_{f_2} = 0.20495 \text{ pF}. \end{aligned}$$

The interconductor contribution to the π-phase capacitance for the structure is $4C_m = 4(0.210\epsilon_0)$. Thus the effective length of the conductor is

$$\begin{aligned} l_{eff} &= 4.45 + \frac{0.20495}{0.43684} - \frac{(0.216)(2.54)(2C_m)}{0.43684} \\ &= 4.87 \text{ cm} \end{aligned}$$

and the corresponding center frequency of the filter is 1.54 GHz.

In conclusion, the present analysis gives a frequency slightly less than that predicted by Nicholson. However, better results will be obtained if the positions of the electric walls can be accurately evaluated, especially at the shorted end of the conductor. The effective length concept has been emphasized because it can be conveniently used in other applications of such structures. Furthermore, it can be easily improved upon by analog or rigorous analytic techniques based on conformal transformation methods.

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Simplifying Maxwell's Equations in Gyrotropic Media

The purpose of this correspondence is to describe a method of handling Maxwell's equations in gyrotropic media. We have found this method to be particularly useful in the analysis of a small, ferrite filled waveguide, but believe it may be useful in general. The form of the equations appear to be considerably less complicated and capable of affording more insight than the methods employed in the literature [1]–[3]. We will make use of a pair of oppositely rotating, elliptically polarized vectors which, as is well known [4] can be used to diagonalize parts of the permeability, permittivity, and conductivity tensors. However, except in infinite media, these polarized variables will not simplify Maxwell's equations unless a new formulation is used.

For illustrative purposes, a gyromagnetic medium will be considered. The medium is taken to be magnetized in the z direction and a Cartesian coordinate system assumed. The time dependence and z variation are taken to be $e^{j\omega t}$ and $e^{-j\beta z}$, respectively. Only the magnetic equation of motion (permeability tensor) and Maxwell's two curl equations are required, as the divergence relations are redundant.

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DEFINITIONS

Let

$$b^\pm = b_x \pm jb_y.$$

Therefore,

$$b_x = \frac{b^+ + b^-}{2}; \qquad b_y = \frac{b^+ - b^-}{2j}.$$

Similar definitions are used for: $e^\pm$, $h^\pm$, e_x, e_y, h_x, h_y. We shall define two operators of the form

$$\nabla^\pm = \frac{\partial}{\partial x} \pm j\frac{\partial}{\partial y}.$$

Note

$$\begin{aligned} \nabla^+\nabla^- &= \nabla^-\nabla^+ = \nabla_t^2 \\ &= \text{transverse divergence gradient.}^1 \end{aligned}$$

MAGNETIC EQUATIONS OF MOTION

The exchange-free, lossless, magnetic equation of motion can be manipulated to give the following set of equations [4]:

$$\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} \mu_{11} & j\mu_{12} & 0 \\ -j\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}. \tag{1}$$

Changing to polarized variables:

$$\begin{aligned} b^+ &= \mu^+ h^+ \\ b^- &= \mu^- h^- \\ b_z &= \mu_0 h_z \end{aligned} \tag{2}$$

where:

$$\frac{\mu_{11}}{\mu_0} = 1 + \frac{\omega_m\omega_0}{\omega_0^2 - \omega^2}; \qquad \frac{\mu_{12}}{\mu_0} = \frac{\omega_m\omega}{\omega_0^2 - \omega^2}$$

$$\begin{aligned} \mu^+ &= \mu_{11} + \mu_{12} = \mu_0\left(1 + \frac{\omega_m}{\omega_0 - \omega}\right) \\ \mu^- &= \mu_{11} - \mu_{12} = \mu_0\left(1 + \frac{\omega_m}{\omega_0 + \omega}\right) \\ \omega_m &= \gamma 4\pi M_s \\ \omega_0 &= \gamma H_{DC} \\ \gamma &= \text{gyromagnetic ratio} \\ 4\pi M_s &= \text{saturation magnetization} \\ H_{DC} &= \text{magnetic field present in gyrotropic medium} \\ \mu_0 &= \text{permeability of free space.} \end{aligned}$$

MAXWELL'S EQUATIONS

Due to the similarity of the curl equations, it is convenient to perform parallel operations as follows:

$$\nabla \times \boldsymbol{e} = -j\omega \boldsymbol{b} \qquad \nabla \times \boldsymbol{h} = j\omega\epsilon \boldsymbol{e}. \tag{3}$$

In matrix form

$$\begin{bmatrix} 0 & j\beta & \frac{\partial}{\partial y} \\ -j\beta & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = -j\omega \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \Bigg\| \begin{bmatrix} 0 & j\beta & \frac{\partial}{\partial y} \\ -j\beta & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = j\omega\epsilon \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}. \tag{4}$$

[1] Similar results can be obtained for cylindrical coordinates in either of two ways. The first is by defining new variables like $b^\pm = b_r \pm jb_\phi$ and $\nabla^\pm = \partial/\partial r \pm j(1/r)\partial/\partial\phi$. With these definitions, the transverse divergence gradient, ∇_t^2 becomes $(1/r + \nabla^-)\nabla^+ = (1/r + \nabla^+)\nabla^-$. The second is by preserving the Cartesian definitions and simply transforming into cylindrical coordinates.

Changing to polarized variables and combining the x and y component equations:

$$\beta e^+ - j\nabla^+ e_z = -j\omega b^+ \quad \Big\| \quad \beta h^+ - j\nabla^+ h_z = +j\omega\epsilon e^+ \tag{5}$$

$$-\beta e^- + j\nabla^- e_z = -j\omega b^- \quad \Big\| \quad -\beta h^- + j\nabla^- h_z = +j\omega\epsilon e^-. \tag{6}$$

In order to proceed further, it is necessary to split e_z and h_z into two parts as follows:

$$e_z = \frac{e_z^+ + e_z^-}{2}; \qquad h_z = \frac{h_z^+ + h_z^-}{2}; \qquad b_z^+ = \mu_0 h_z^+; \qquad b_z^- = \mu_0 h_z^-.$$

The defining equations for $e_z^\pm$ and $h_z^\pm$ are derived from the z components of the curl equations. After some algebra we can let:

$$-j\omega b_z^+ = -j\nabla^- e^+ \quad \Big\| \quad j\omega\epsilon e_z^+ = -j\nabla^- h^+ \tag{7}$$

$$-j\omega b_z^- = +j\nabla^+ e^- \quad \Big\| \quad j\omega\epsilon e_z^- = +j\nabla^+ h^-. \tag{8}$$

Without performing the above steps we found it impossible to proceed further. If we now combine (7) and (8) with (2), (5), and (6), $e^\pm$ and $h^\pm$ can be eliminated and the following equations derived:

$$e_z^+ = -j\frac{\omega\mu_0}{\beta} h_z^+ - \frac{j\nabla_t^2}{\beta\omega\epsilon} h_z \quad \Big\| \quad h_z^+ = j\frac{\omega\epsilon\mu^+}{\beta\mu_0} e_z^+ + \frac{j\nabla_t^2}{\omega\beta\mu_0} e_z \tag{9}$$

$$e_z^- = +j\frac{\omega\mu_0}{\beta} h_z^- + \frac{j\nabla_t^2}{\beta\omega\epsilon} h_z \quad \Big\| \quad h_z^- = -j\frac{\omega\epsilon\mu^-}{\beta\mu_0} e_z^- - \frac{j\nabla_t^2}{\omega\beta\mu_0} e_z \tag{10}$$

or in matrix form

$$\begin{bmatrix} e_z^+ \\ e_z^- \end{bmatrix} = \begin{bmatrix} -\frac{j\omega\mu_0}{\beta} - \frac{j\nabla_t^2}{2\beta\omega\epsilon} & -\frac{j\nabla_t^2}{2\beta\omega\epsilon} \\ +\frac{j\nabla_t^2}{2\beta\omega\epsilon} & +\frac{j\omega\mu_0}{\beta} + \frac{j\nabla_t^2}{2\beta\omega\epsilon} \end{bmatrix} \begin{bmatrix} h_z^+ \\ h_z^- \end{bmatrix} \quad \Big\| \quad \begin{bmatrix} h_z^+ \\ h_z^- \end{bmatrix} = \begin{bmatrix} \frac{j\omega\epsilon\mu^+}{\beta\mu_0} + \frac{j\nabla_t^2}{2\omega\beta\mu_0} & +\frac{j\nabla_t^2}{2\omega\beta\mu_0} \\ -\frac{j\nabla_t^2}{2\omega\beta\mu_0} & -\frac{j\omega\epsilon\mu^-}{\beta\mu_0} - \frac{j\nabla_t^2}{2\omega\beta\mu_0} \end{bmatrix} \begin{bmatrix} e_z^+ \\ e_z^- \end{bmatrix}. \tag{11}$$

Note if we add (9) and (10)

$$e_z = -\frac{j\omega\mu_0}{2\beta}(h_z^+ - h_z^-) \quad \Big\| \quad h_z = \frac{j\omega\epsilon}{2\beta\mu_0}(\mu^+ e_z^+ - \mu^- e_z^-). \tag{12}$$

The above equations are complete and can be used to solve boundary value problems. By further manipulation of (9) and (10), $e_z^\pm$ and $h_z^\pm$ can be combined and the two coupled wave equations, first derived by Kales [1] are arrived at:

$$\nabla_t^2 h_z + \left(\omega^2\epsilon\mu_0 - \frac{2\beta^2\mu_0}{\mu^+ + \mu^-}\right) h_z = -j\omega\epsilon\beta\left(\frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}\right) e_z \tag{13}$$

$$\nabla_t^2 e_z + \left(\frac{2\omega^2\epsilon\mu^+\mu^-}{\mu^+ + \mu^-} - \beta^2\right) e_z = +j\omega\beta\mu_0\left(\frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}\right) h_z \tag{14}$$

or

$$\nabla_t^2 h_z + \left(\omega^2\epsilon\mu_0 - \frac{\beta^2\mu_0}{\mu_{11}}\right) h_z = -j\omega\epsilon\beta\frac{\mu_{12}}{\mu_{11}} e_z \tag{15}$$

$$\nabla_t^2 e_z + \left(\omega^2\epsilon\frac{\mu_{11}^2 - \mu_{12}^2}{\mu_{11}} - \beta^2\right) e_z = +j\omega\beta\mu_0\frac{\mu_{12}}{\mu_{11}} h_z. \tag{16}$$

To illustrate the potential utility of this method, we can consider wave propagation in an undersized waveguide completely filled with a longitudinally magnetized, lossless ferrite. The exact solution for this problem is quite complicated [2], [6]. However, a very accurate solution, valid over a wide band of frequencies above ferromagnetic resonance, can be derived with one simple approximation.

The approximation to be made is not apparent in the form of Kales' equations (13) and (14), but by careful examination of (11) (left column) one can readily see that when the transverse variation is much greater than the dielectric wavenumber ($\nabla_t^2 \gg \omega^2\mu_0\epsilon$), $e_z^+ \doteq -e_z^-$ and $e_z \doteq 0$ (providing $h_z \neq 0$). If we set e_z to zero, (13) indicates a propagating TE wave will result. This mode will satisfy all the boundary conditions with the exception of $e_z = 0$ on the walls. Due to the birefringent nature of the material, another propagating mode is also possible. Equation (15) indicates this mode will propagate (h_z small) if there is a rapid transverse decay. Thus the exact solution may be viewed as a large amplitude TE mode which fills the guide and a small amplitude "TM" mode which clings to the walls. Both modes must have the same axial propagation constant. For most practical applications, only the TE mode is important.

It is interesting to note that the mode which fills the guide is the same mode that results from the magnetostatic approximation [7], [8] in which the curl h is set equal to zero.

The solution as obtained with the above theory is more satisfying than that of the magnetostatic approximation for several reasons. First, the other mode is completely missing from the magnetostatic solution. Second, while the spatial variation of h is much greater than the time variation of the displacement current, the ratio of e^+/h^+ is not negligible and, in fact, can become extremely large near resonance. Finally, there is a tendency to regard the magnetostatic mode as a separate entity, while in fact it is simply one of the electromagnetic modes. When the spin exchange forces are neglected, as above, the dispersion relation has only two roots, and the magnetostatic dispersion relation corresponds to a portion of one of these roots near resonance.

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